# Behaviour of Lagrangian Approximations in Spherical Voids

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1 February 2008



**ABSTRACT**
We study the behaviour of spherical Voids in Lagrangian perturbation theories L(n), of which the Zel'dovich approximation is the lowest order solution L(1). We find that at early times higher order L(n) give an increasingly accurate picture of Void expansion. However at late times particle trajectories in L(2) begin to turnaround and converge leading to the *contraction* of a Void, a sign of pathological behaviour. By contrast particle trajectories in L(3) are well behaved and this approximation gives results in excellent agreement with the exact top-hat solution as long as the Void is not too underdense. For very underdense Voids, L(3) evacuates the Void much too rapidly leading us to conclude that the Zel'dovich approximation L(1), remains the best approximation to apply to the late time study of Voids. The behavior of high order approximations in spherical voids is typical for asymptotic series and may be generic for Lagrangian perturbation theory.

**Key words:** cosmology, large-scale structure of the Universe: voids.

## 1 INTRODUCTION

It is widely felt that gravitationally bound systems such as galaxies and groups and clusters of galaxies arose due to the gravitational amplification of small density fluctuations already existing at the time of last scattering (and reflected in the angular fluctuations in the CMBR measured by the COBE satellite).

The growth of density perturbations in a homogeneous and isotropic Universe can be studied in essentially two distinct ways. In the *Eulerian* approach (pioneered by Jeans and Lifshitz), solutions to the Euler-Poisson system are sought by means of a dimensionless density perturbation $\delta(\mathbf{x},t) = [\rho(\mathbf{x},t) - \rho_0]/\rho_0$, which is expanded in a perturbation series having the form

$$\delta(\mathbf{x},t) = \sum_{n=1}^{\infty} \delta^{(n)}(\mathbf{x},t) = \sum_{i=1}^{\infty} D_+^n(t)\Delta^{(n)}(\mathbf{x}) \tag{1}$$

in a spatially flat matter dominated Universe (similar expansions are made for the velocity field and gravitational potential). $\delta^{(1)}$ corresponds to the linear theory limit for which $D_+(t) \propto a(t) \propto t^{2/3}$. Eulerian perturbation theory can be successfully applied as long as the series (1) converges *i.e.* $|\delta| << 1$.

An alternate viewpoint to Eulerian theory was proposed by Zel'dovich who suggested perturbing particle trajectories as a means to understand gravitational instability. In a general *Lagrangian* framework

$$\mathbf{x} = \mathbf{q} + \mathbf{\Psi}(t,\mathbf{q}) \tag{2}$$

where $\mathbf{x}$ and $\mathbf{q}$ are the Eulerian and Lagrangian coordinates of a particle/fluid element. The displacement vector $\mathbf{\Psi}$ is related to the initial velocity field and its derivatives, in Zeldovich's approximation $\mathbf{\Psi} = \mathbf{\Psi}^{(1)} = D_+(t)\psi_i^{(1)}(\mathbf{q})$, $\psi_i^{(1)}$ being the initial velocity field. In recent years attempts have been made to consider higher order corrections to the Zel'dovich approximation by treating $\mathbf{\Psi}$ as an expansion (Moutarde et al. 1991, Bouchet et al. 1992, Buchert 1993, Catelan 1995, Bouchet et al. 1995)

$$\mathbf{\Psi} = \sum_{i=1}^{n} \mathbf{\Psi}^{(n)} \tag{3}$$

Expansions of the form (2), (3) are frequently referred to as *n-th order* Lagrangian perturbation series L(n), the first order series L(1) being the Zel'dovich approximation. One might expect (in analogy with Eulerian perturbation theory – E(n)) L(n) to get more accurate with increasing 'n' as higher order terms are included in $\Psi$ (Coles & Sahni 1995). This is indeed the case during the weakly non-linear regime as demonstrated by Munshi, Sahni & Starobinsky (1994) who showed that results derived from L(n) matched those derived from E(n) as long as the system is weakly nonlinear. An important issue which has so far remained largely unaddressed concerns the domain of convergence of the L(n) series (2) & (3). Although several recent studies have demonstrated the increasing accuracy of higher order L(n), most

treatments (with the exception of Bouchet et al. 1995) simulated overdense regions and left unanswered the related (and important) issue of the accuracy of L(n) in underdense regions (Voids). We shall address this issue in this letter by studying the behavior of L(n) in spherical underdense regions complementing and extending the analysis of Munshi et al. (1994) for overdense regions. (We leave a more general analysis treating generic initial conditions to a later work (Buchert, Sahni & Shandarin 1995)).

## 2 LAGRANGIAN PERTURBATION THEORY IN VOIDS

In a spatially flat, matter dominated Universe $\Psi_i^{(n)}$ factorise:

$$\Psi_i^{(n)} = D_+^n(t)\psi_i^{(n)}(\mathbf{q}), \qquad D_+(t) = \frac{3t^2}{2a^2} \propto a(t) \propto t^{2/3} \tag{4}$$

Furthermore higher orders in $\mathbf{\Psi}^{(n)}$ $(n > 1)$ can be constructed from lower orders by means of an iterative procedure (Moutarde et al. 1991, Buchert 1992, Lachieze-Rey 1993), so that $\psi_i^{(1)} = -\partial\phi_0(\mathbf{q})/\partial q_i$, for L(1) (the Zel'dovich approximation), for L(2) (equivalently the *post-Zel'dovich approximation*) (Bouchet et al. 1992, Buchert 1992)

$$\psi_{i,i}^{(2)} = -\frac{3}{14}\left((\psi_{i,i}^{(1)})^2 - \psi_{i,j}^{(1)}\psi_{j,i}^{(1)}\right) \tag{5}$$

$$\psi_{i,j}^{(2)} = \psi_{j,i}^{(2)} \tag{6}$$

For L(3):

$$\begin{aligned}\psi_{i,i}^{(3)} &= -\tfrac{5}{9}(\psi_{i,i}^{(2)}\psi_{j,j}^{(1)} - \psi_{i,j}^{(2)}\psi_{j,i}^{(1)}) - \tfrac{1}{3}det[\psi_{i,j}^{(1)}],\\ \psi_{i,j}^{(3)} - \psi_{j,i}^{(3)} &= \tfrac{1}{3}(\psi_{i,k}^{(2)}\psi_{k,j}^{(1)} - \psi_{j,k}^{(2)}\psi_{k,i}^{(1)})\end{aligned} \tag{7}$$

Here coma implies partial derivative with respect to $\mathbf{q}$, summation over repeated indices is assumed (cf. Juszkiewicz et al. 1993, Bernardeau 1993).

Let us consider the case of a spherical top-hat underdensity for which $\delta^{(1)}(\mathbf{x},t) = -a(t)$, the initial gravitational potential can be obtained from the Poisson equation and has the time independent form (Munshi et al. 1994) $\phi_0 = a^2r_0^2\delta/9t^2 = -a^3r_0^2/9t^2$.

The exact solution for the expansion of a top-hat Void can be parametrised as

$$\begin{aligned}R(\theta) &= (3/10)(\cosh\theta - 1)\\ a(\theta) &= (3/5)\left[(3/4)(\sinh\theta - \theta)\right]^{2/3}\end{aligned} \tag{8}$$

Here $R = a(t)r/r_0$ is the physical (and $r$ the comoving) particle trajectory. Knowing $r(a)$ the equation of mass conservation allows one to determine the density contrast

$$\delta = (r_0/r)^3 - 1 \tag{9}$$

which for negative density fluctuations has the form

$$\delta_{TH}(\theta) = \frac{9}{2}\frac{(\theta - \sinh\theta)^2}{(\cosh\theta - 1)^3} - 1 \tag{10}$$

The related dimensionless peculiar velocity $(1/aH)\ div_{\mathbf{x}}\mathbf{u} = \theta$ ($\mathbf{u} = a\dot{\mathbf{x}}$, $\mathbf{x}$ is the comoving coordinate) may be determined from the Eulerian mass conservation equation (Munshi et al. 1994)

$$a\frac{d}{da}\delta + (1+\delta)\theta = 0 \tag{11}$$

Particle trajectories for spherical top-hat expansion are easy to derive in L(n), substituting $\phi_0 = -a^3r_0^2/9t^2$ in (6) & (7) we get

$$\begin{aligned}r_1(a) &= r_0(1 + \frac{a}{3})\\ r_2(a) &= r_0\left(1 + \frac{a}{3} - \frac{a^2}{21}\right)\\ r_3(a) &= r_0\left(1 + \frac{a}{3} - \frac{a^2}{21} + \frac{23a^3}{1701}\right)\end{aligned} \tag{12}$$

$r_n$ is the particle trajectory in L(n). From (12) and the conservation condition (9) we obtain expressions for the density in a top-hat void

$$\begin{aligned}\delta_1 &= \left(1 + \frac{a}{3}\right)^{-3} - 1\\ \delta_2 &= \left(1 + \frac{a}{3} - \frac{a^2}{21}\right)^{-3} - 1\\ \delta_3 &= \left(1 + \frac{a}{3} - \frac{a^2}{21} + \frac{23a^3}{1701}\right)^{-3} - 1\end{aligned} \tag{13}$$

The related expressions for the velocity field are determined from (11)

$$\begin{aligned}\theta_1 &= a\left(1 + \frac{a}{3}\right)^{-1}\\ \theta_2 &= (a - \frac{2}{7}a^2)/(1 + \frac{a}{3} - \frac{a^2}{21})\\ \theta_3 &= (a - \frac{2}{7}a^2 + \frac{23}{189}a^3)/(1 + \frac{a}{3} - \frac{a^2}{21} + \frac{23}{1701}a^3)\end{aligned} \tag{14}$$

(The expressions for a top-hat overdensity can be obtained from (12) – (14) by the transformation $a \to -a$.)

Our results for the evolution of overdense and underdense regions are summarised in figures (1) & (2) for top-hat density and velocity fields respectively. Looking at the right-hand side of figure (1) (and the left hand side of figure (2)), we see that higher order L(n) outperform lower orders in matching the results of exact top-hat *collapse*. The remaining two panels – showing top-hat Void *expansion*, however give a strikingly different picture. We find that although L(2) and L(3) do paint an accurate picture at early times, their accuracy declines once the top-hat density falls below $\delta_{TH} \approx -0.7$. In fact, for L(2) particle trajectories begin to converge (rather than diverge) at late times, leading to the 'collapse' of a top-hat Void, a rather pathalogical result ! (also see Bouchet et al. 1995). Third order fares considerably better than second order since particle trajectories always diverge leading to Void expansion. However, as we see from figures (1) and (2), the third order Void expands too rapidly at late times, causing the asymptotic limit $\delta \to -1$ to be reached much too early. (Interestingly $\delta_2(a) > \delta_{TH}(a)$, whereas $\delta_1(a), \delta_3(a) < \delta_{TH}(a)$.) (In simulations with random initial conditions one might expect shell crossing to occur before the density contrast in Voids has dropped to $\delta_{TH} \approx -0.7$, as a result the pathological behaviour of L(2) may not be discernable in simulations which evolve particles till the epoch of shell crossing but no further (Buchert, Melott & Weiss 1994).)

Although the above discussion has been limited to homogeneous Voids, our main conclusions appear to be

more general. To demonstrate this we consider the case of a spherical density perturbation with an *arbitrary* initial potential $\phi_0(r_0)$. Expressions for L(1), L(2) and L(3) turn out to be remarkably simple: $\vec{\psi}_i^{(1)} = -(\phi_0'/r_0)\ \vec{r}_0$, $\vec{\psi}_i^{(2)} = -3/7(\phi_0'/r_0)^2\ \vec{r}_0$, $\vec{\psi}_i^{(3)} = -23/63(\phi_0'/r_0)^3\ \vec{r}_0$, where $\phi_0' = \partial\phi_0/\partial r_0$. As a result

$$\vec{r_1} = \vec{r_0}\Big[1 - (D_+\phi_0'/r_0)\Big]$$

$$\vec{r_2} = \vec{r_0}\Big[1 - (D_+\phi_0'/r_0) - \frac{3}{7}(D_+\phi_0'/r_0)^2\Big]$$

$$\vec{r_3} = \vec{r_0}\Big[1 - (D_+\phi_0'/r_0) - \frac{3}{7}(D_+\phi_0'/r_0)^2 - \frac{23}{63}(D_+\phi_0'/r_0)^3\Big] \quad (15)$$

(Related expressions for the density and velocity fields can be easily determined from (9) and (11).) Specialising to power law potentials $\phi_0 = -Ar_0^n$, $A = a^3/9t^2 = constant$, we get

$$\vec{r_2} = \vec{r_0}\Big[1 + nD_+Ar_0^{n-2} - \frac{3}{7}(nD_+Ar_0^{n-2})^2\Big] \quad (16)$$

$(D_+A = a/6)$. We find that for L(2) the displacement vector's $\vec{\psi}_i^{(2)}$ and $\vec{\psi}_i^{(1)}$ have opposite directions leading to expansion at early times being replaced by collapse at later epochs. The form of (5), (6) suggests that this behavior may be generic. Indeed the transformation $\phi_0 \to -\phi_0$ (transforming a concave potential into a convex one) results in $\vec{\psi}_i^{(1)} \to -\vec{\psi}_i^{(1)}$, $\vec{\psi}_i^{(2)} \to \vec{\psi}_i^{(2)}$, the fact that the displacement field $\vec{\psi}_i^{(2)}$ remains inwardly directed regardless of the sign of the potential, results in particles flowing inwards at late times leading to the eventual 'collapse of Voids' in L(2). Thus we find that although L(2) and L(3) start out being more accurate than L(1) at the commencement of expansion, their accuracy declines with time, and somewhat surprisingly, L(1) provides a more accurate picture of Void expansion than either L(2) or L(3) at very late times (see figures (1) & (2)).

It would be interesting to determine whether the pathalogical behaviour of L(2) afflicts all even orders of the Lagrange perturbation series or whether it is restricted to second order alone. Although a completely general treatment of this kind lies beyond the scope of the present letter, it is possible to assess what happens *at any arbitrary order* for the case of a top-hat Void. This case is exactly solvable since an expression for the particle trajectory in L(n) can be found by expanding the exact top-hat solution (8) in powers of $a$ (Munshi et al. 1994), leading to

$$\vec{r_n} = \vec{r_0}\Big[1 - \sum_{i=1}^{n}(-1)^i\alpha_i a^i\Big] \quad (17)$$

($\alpha > 0$; for $n = 5$, $\alpha_1 = 1/3$, $\alpha_2 = 1/21$, $\alpha_3 = 23/1701$, $\alpha_4 = 1894/392931$, $\alpha_5 = 3293/1702701$.) The form of (17) indicates that all even order terms will have negative signs leading to the eventual contraction of top-hat Voids in L(n=even). Odd orders are better than even orders but overestimate Void expansion at late times giving rise to very empty Voids as demonstrated in figures (1) & (2) for L(1) & L(3).

## 3 CONCLUSIONS

We have discussed the evolution of spherical Voids in the framework of Lagrangian Perturbation series L(n) of which the Zel'dovich approximation is the first order solution. We find that L(n) with n-even overestimate the density in Voids, whereas L(n) with n-odd underestimate it. On the whole we find that L(n=odd) provide better descriptors of Void expansion than L(n=even). The Zel'dovich solution L(1) outperforms L(3), L(5),... at late times which is typical for asymptotic or semiconvergent series. We have not proved it but speculate that this may be a generic property of the Lagrangian and probably Eulerian perturbative theories of gravitational instability.

It should be mentioned that the relative accuracy of L(1) in spherical top-hat Voids has earlier been tested against the accuracy of several Eulerian approximations (EA) by Sahni & Coles (1995) and Bouchet et al. (1995), who demonstrated that L(1) performed significantly better than EA at all times. The results of this letter strengthen that conclusion and show that the Zel'dovich approximation is the best non-linear approximation to apply to the late time study of spherical Voids (also see Sahni, Sathyaprakash & Shandarin 1994).

**Acknowledgments:** The authors acknowledge useful discussions with Thomas Buchert, Dipak Munshi and Alexei Starobinsky. Acknowledgements are also due to the Smithsonian Institution, Washington, USA, for International travel assistance under the ongoing Indo-US exchange program at IUCAA, Pune, India. S. Shandarin acknowledges the support from NSF Grant AST-9021414, NASA Grant NAGW-3832 and a University of Kansas GRF 95 Grant.

## REFERENCES

Bernardeau F., 1992, ApJ, 392, 1.
Bouchet F.R., Juszkiewicz R., Colombi S., Pellat R., 1992, ApJ, 394, L5
Bouchet F.R., Colombi, S., Hivon, E., Juszkiewicz R., 1995, *Astron. Astrophys.*, 296, 575
Buchert, T., 1992, *Astron. Astrophys.*, 223, 9
Buchert, T., 1993, *Astron. Astrophys.*, 267, l51
Buchert, T., Melott, A.L., Weiss, A.G., 1994, *Astron. Astrophys.*, 288, 349
Buchert, T., Sahni, V., Shandarin, S.F., 1995, In preparation.
Catelan, P., 1995, MNRAS, 276, 115
Coles, P., Sahni, V., 1995, The Observatory, In press.
Juszkiewicz R., Bouchet F.R., Colombi S., 1993, ApJ, 412, L9
Lachieze–Rey M., 1993, ApJ, 408, 403
Moutarde, F.,J. Alimi J.M., Bouchet F.R., Pellat R., Ramani A., 1991, ApJ, 382, 377
Munshi, D., Sahni, V., Starobinsky, A.A., 1994, ApJ, 436, 517
Sahni, V., Sathyaprakash, B.S., Shandarin, S.F., 1994, ApJ, 431, 20.
Sahni, V., Coles, P., 1995, Physics Reports, In press.
Shandarin, S.F., Zel'dovich, Ya. B., 1989, Rev. Mod. Phys., 61, 185
Zel'dovich, Ya.B., 1970, A&A, 4, 84

**Figure 1.** The density contrast $\delta_{APP}$ in Lagrangian perturbation series L(n) is shown plotted against the exact top-hat solution $\delta_{EX}$ for underdense regions (lower left) and overdense regions (upper right). We find that whereas the accuracy of L(n) increases with 'n' when describing the behaviour of overdense regions, L(n) with $n > 1$, do not fare as well when applied to underdense regions or 'Voids'. For Voids we notice that although L(n) with $n = 2, 3$ are initially more accurate than L(1) (Zel'dovich approximation), their accuracy worsens with time. We also find that L(2) shows pathalogical behaviour at late times when $\delta_{EX} < -0.7$. (Note: Different scaling has been used to label the negative and positive axes.)

**Figure 2.** The dimensionless velocity field $\theta_{APP}$ in Lagrangian perturbation series L(n) is shown plotted against the exact solution $\theta_{EX}$, for overdense regions (lower left) and underdense regions (upper right). We find that L(n), $n = 2, 3$ give better results than L(1) for overdense but not for underdense regions. (Note: Different scaling has been used to label the negative and positive axes.)